The Generation of Rating Curves from Data

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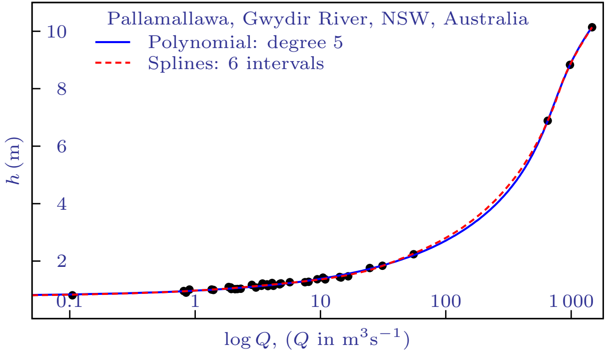
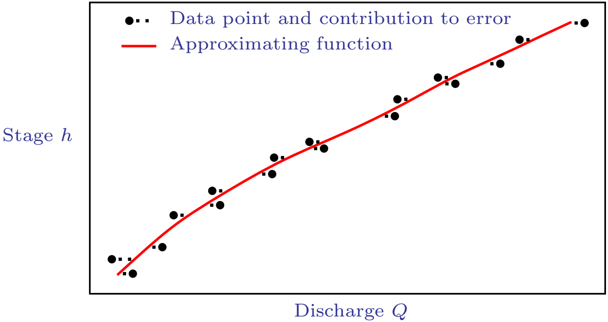


It doesn't *look* like Rocket Science, but there are some problems to overcome

Traditional view of rating curve approximation

- The power function
$$Q = C\,(h - h_0)^{\mu}\ ,$$
where C, h_0 and μ are constants, and which is a straight line on $(\log Q, \log(h - h_0))$ axes.
- On one hand this is too simple, with only three parameters, and is limited in its accuracy and generality.
- On the other hand, it is too complicated, such that the three parameters occur nonlinearly and solving for them is difficult such that visual/manual methods are often used.
- We want to automate and generalise the operation.

Least-squares approximation



There are various problems

Problem 1 – Rapid variation (large curvature) of Q with h at small discharges

- Where there is *Local Control* for low flows, this often looks like the power function $Q = C\,(h - h_0)^{\mu}$ where $\mu = 1.5 \ldots 2.5$.
- Rewriting this with $\nu = 1/\mu$ as

$$Q^{\nu} = C^{\nu}\,(h - h_0) = a_0 + a_1 h \;,$$

  which is linear in the parameters a_0 and a_1, and is a good approximation for $h - h_0$ small. The curvature has been taken into the Q^{ν} term.
- Fenton and Keller (2001, §6.3.2) suggested the simple generalisation:

$$Q^{\nu} = a_0 + a_1 h + a_2 h^2 + \ldots + a_M h^M \;.$$

- They recommended a value of $\nu = \frac{1}{2}$, the mean value in hydraulic discharge formulae for a sequence of weir and channel cross-sections that modelled local and channel control.
- They calculated and presented one result – where it worked well.

Various problems (cont.)

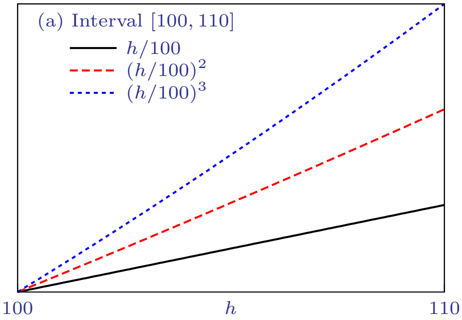
Problem 2 – the range of discharge is huge

- Q can vary by a factor of 10 000, for example between $Q \approx 1$ to $10^4\,\mathrm{m}^3\,\mathrm{s}^{-1}$, with the upper limit worse for Australians (ML/d in the water industry) and USAmericans, ($\mathrm{ft}^3\mathrm{s}^{-1}$)
- In fact, we have already solved the problem by using Q^ν with $\nu \approx 1/2$, giving a range to approximate with a factor of only $10\,000^{1/2} = 100$.

Various problems (cont.)

Problem 3 – Polynomials in stage h can have huge problems

- The previous problems were almost obvious and the solution almost obvious. The next problem is more subtle but can be much more serious.
- Consider representing a rating curve with actual elevation as stage, maybe between $h = 100\,\mathrm{m}$ and $h = 110\,\mathrm{m}$, and represented by a cubic function $Q^\nu = a_0 + a_1 h + a_2 h^2 + a_3 h^3$. Let us look at the individual monomials h^m on that interval (plotting with h horizontal and scaling each by 100^m):

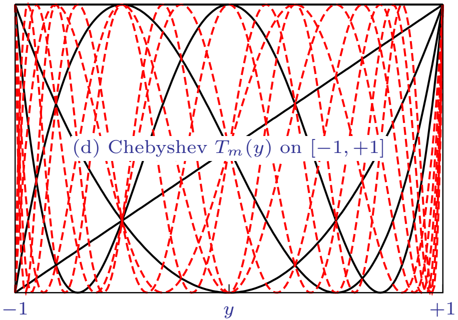


- The monomials all look like straight lines! Consider the problem of approximating a rating curve with finite curvature – the coefficients a_1 *etc* would have to work very hard (being large, and oscillating in sign).

Various problems (cont.)

Overcoming Problem 3

- The answer to the problem is to use Chebyshev polynomials T_m, where every one looks different from every other one, and so they can approximate almost anything efficiently:



Various problems (cont.)

Problem 4 – the least-squares equations are badly conditioned

- The unknown coefficients a_m are found by least-squares fitting to N data points (h_n, Q_n^ν) for $n = 1, \ldots, N$ such that the mean-square error of the approximating function over all the points is minimised

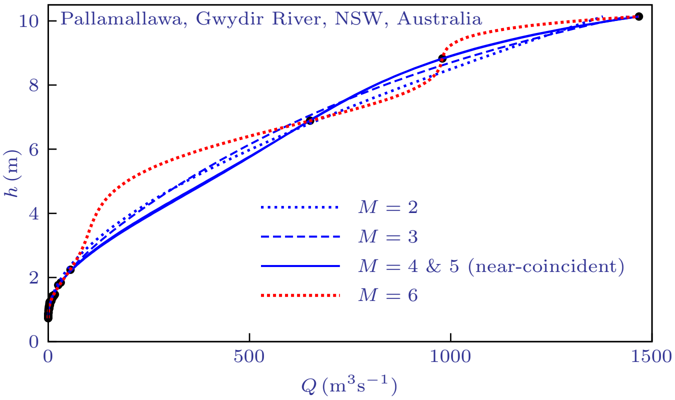
$$\varepsilon = \sum_{n=1}^{N} \left(a_0 + a_1 T_1(h_n) + \ldots + a_M T_M(h_n) - Q_n^\nu\right)^2$$

- The traditional approach (differentiate with respect to each a_m, equate to zero and solve the M equations) can be badly conditioned.
- The solution is to use optimisation methods, searching for a minimum in $(M+1)$-dimensional space. Package software works well.

Various problems (cont.)

Problem 5 – How many terms to include in the polynomial?

- With increasing number of terms M, *sooner or later the degree of approximation becomes too high and unacceptable oscillations appear*



Experience

- All the above measures were implemented by Fenton (2015, 2018) with quite satisfactory results for 8 different stations.
- McMahon and Peel (2019) then used them to obtain 622 (!!!) rating curves from 171 Australian Bureau of Meteorology Hydrologic Reference Stations. They too found that the methods worked well – with the exception of about 0.5% of the stations, where there was difficulty approximating the low-flow data.
- For the development of a stand-alone computer program the problems of occasional unusual low-flow data and automatically determining the level of approximation were not solved.
- Leading to the next method ...

Overcoming remaining problems – Approximating Splines – 1

- Instead of using *global* approximation (the same function for all the data), a different approach is to use *piecewise-continuous* approximation in the form of quadratic functions, where each approximates just part of the range of data, but which is required to merge smoothly with its neighbours.
- This is more flexible in handling unusual low-flow data and is much less-sensitive to the level of approximation – it never goes dramatically wrong as we saw for polynomial approximation.

Approximating Splines – 2

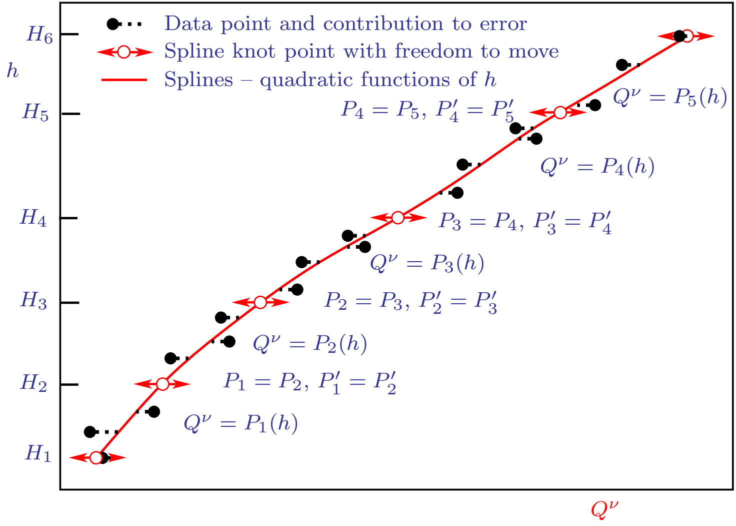
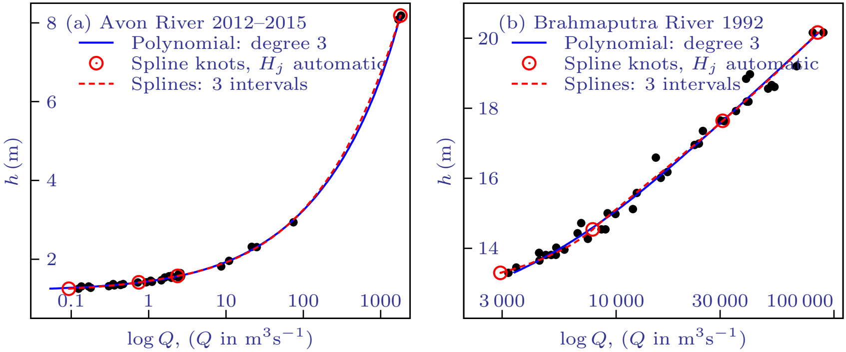
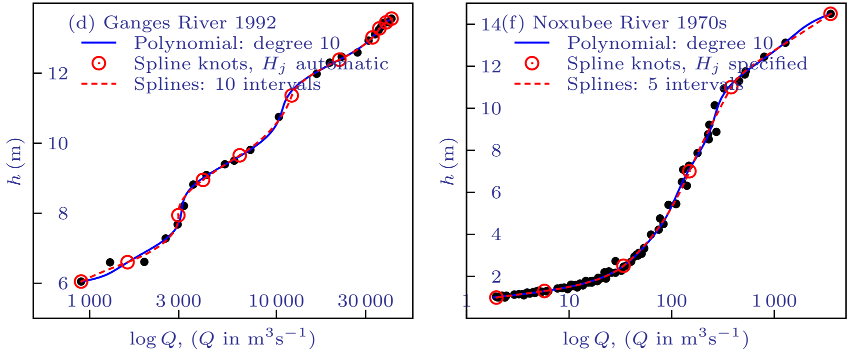


Figure: Spline approximation of rating points, showing how knot points can move so as to minimise the total error, maintaining continuity of function and slope across each knot point

Examples - 1 & 2


(a) Avon River 2012–2015
Polynomial: degree 3
Spline knots, H_j automatic
Splines: 3 intervals
h (m)
8
6
4
2
0.1
1
10
100
1000
log Q, (Q in m^3s^{-1})
(b) Brahmaputra River 1992
Polynomial: degree 3
Spline knots, H_j automatic
Splines: 3 intervals
h (m)
20
18
16
14
3 000
10 000
30 000
100 000
log Q, (Q in m^3s^{-1})

Examples - 3 & 4

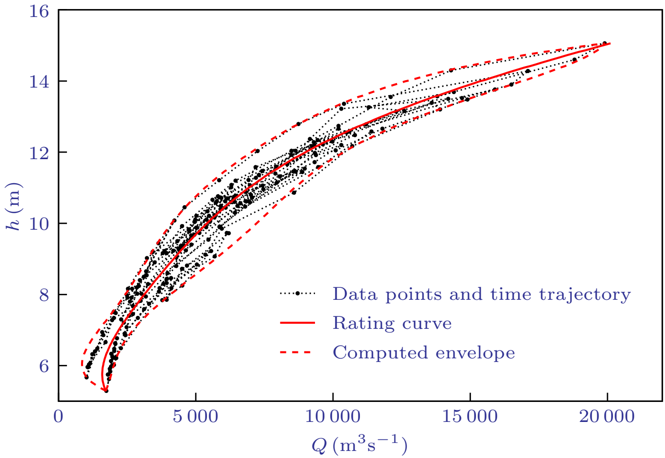


(d) Ganges River 1992
Polynomial: degree 10
Spline knots, H_j automatic
Splines: 10 intervals
h (m)
6
8
10
12
1 000
3 000
10 000
30 000
$\log Q$, (Q in m^3s^{-1})
(f) Noxubee River 1970s
Polynomial: degree 10
Spline knots, H_j specified
Splines: 5 intervals
h (m)
2
4
6
8
10
12
14
1
10
100
1 000
$\log Q$, (Q in m^3s^{-1})

Scattered data 1 – the rating envelope, maxima and minima

- *Short-term* changes in the stream can occur, especially in alluvial streams, where the arrangement of bed grains and possibly bed forms can change almost daily, hence so can resistance *and* the discharge for a given stage.
- This scatter can be incorporated and quantified by the computation of a *Rating Envelope*, so that maximum and minimum expected flows can also be calculated and published.
- The procedure is to follow a succession of steps, eliminating half the data points at each step, all those above/below the latest curve.

Scattered data 2 – the rating envelope, maxima and minima



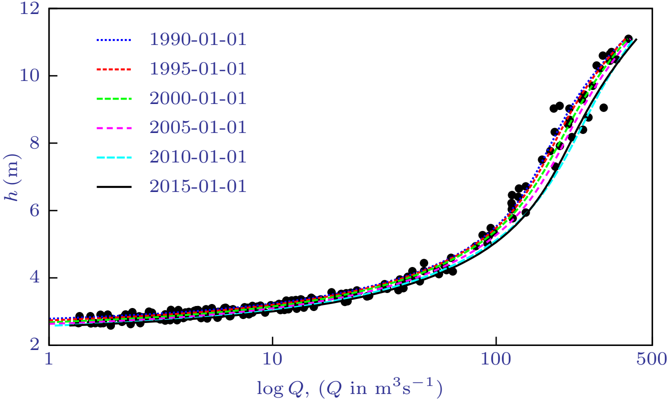
Rating curve and upper and lower envelopes to the data, Station 41 on the Red River, Viet Nam, 1995–1997: four passes of the halving procedure for each of the upper and lower envelopes were applied, so starting with 217 data points, at the end there were about $217/2^4 \approx 15$ for each envelope.

Long-term changes: a rating curve for any day in the past or present

- The importance of each data point can be weighted according to its reliability
- Or, weighted by age so that the oldest points have the smallest contributions.
- Long-term stream changes can be described.
- A rating curve can be constructed for any day, now or in the past.

Example of calculation including age of data points

Calculation of rating curves on specific days, here with a data half-life of 2 years — Noxubee River near Geiger, AL, USA, USGS Station 02448500, from 1984-10-02 to 2015-05-11.



A computer program for calculating rating curves

The speaker has put the spline program on the Web. The program and its operation are described here:

`http://johndfenton.com/Rating-curves/Instructions.pdf`

The program and all files necessary for its operation are in

http://johndfenton.com/Rating-curves/Program-Files.exe

It is necessary to copy that link text and paste it into your Web browser. It is a self-extracting file that, when downloaded and executed (after your computer maybe asks you to say that it is acceptable), unpacks the files, retaining the original file structure, under a directory of your choice. The program file that does the calculations is called, imaginatively, Rating-curve.exe

If anybody has a problem with the program or with a particular site, please write to `johndfenton@gmail.com`

References

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Fenton, J. D. (2018), On the generation of stream rating curves, *Journal of Hydrology* **564**, 748–757.

Fenton, J. D. and Keller, R. J. (2001), The calculation of streamflow from measurements of stage, Technical Report 01/6, Cooperative Research Centre for Catchment Hydrology, Melbourne.
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McMahon, T. A. and Peel, M. C. (2019), Uncertainty in stage-discharge rating curves: application to Australian Hydrologic Reference Stations data, *Hydrological Sciences Journal* **64**(3), 255–275.